MATH 579 Exam 7 Solutions

Part I: C_m is the graph with m vertices and m edges, consisting of a single long cycle. (e.g. C_4 is a square). Recall that a vertex coloring is proper if no two adjacent vertices get the same color. Find the number of proper colorings of this graph with n colors. Simplify your answer.

> Label the edges from $[m]$. Let A_i (for $i \in [m]$) denote those colorings wherein the vertices connected by edge i have the same color. Conveniently, the number of colorings of $A_i \cap A_j$ does not depend on whether i, j are adjacent edges or not. Hence, by inclusion-exclusion, the number of colorings is $n^m - \binom{m}{1}n^{m-1} + \binom{m}{2}n^{m-2} - \binom{m}{3}n^{m-3} + \cdots + (-1)^{m-1}\binom{m}{m-1}n + (-1)^m\binom{m}{m}n$. Apart from the last term (which has n instead of 1), this is exactly $(n-1)^m$, by the binomial theorem. Hence the answer is $(n-1)^m + (-1)^m(n-1)$.

Part II:

1. Calculate $\phi(210)$.

   $\phi(210) = \phi(2)\phi(3)\phi(5)\phi(7) = 1 \cdot 2 \cdot 4 \cdot 6 = 48$.

2. How many n-permutations contain exactly one cycle of length 1?

   There are n ways to choose the cycle; the remainder is a derangement of $n-1$, of which there are D_{n-1}. Hence the answer is $nD_{n-1} = n(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} = n!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}$.

3. How many positive integers are there in $[1000]$ that are neither perfect squares nor perfect cubes?

   Let A_{square} and A_{cube} denote the numbers possessing these two properties. $|A_{square}| = 31$, because $31^2 < 1000 < 32^2$. $|A_{cube}| = 10$, because $10^3 = 1000$. $A_{square} \cap A_{cube}$ are those numbers that are perfect $\text{lcm}(2,3) = 6^{\text{th}}$ powers. There are 3 such, since $3^6 < 1000 < 4^6$. Hence the answer is $1000 - 31 - 10 + 3 = 962$.

4. How many three-digit positive integers are divisible by at least one of six and seven?

   Let A_6 denote the property of being divisible by 6, and A_7 denote the property of being divisible by 7. There are $\lfloor\frac{999}{6}\rfloor = 166$ numbers in $[999]$ divisible by 6, and $\lfloor\frac{99}{6}\rfloor = 16$ numbers in $[99]$ divisible by 6; hence $166 - 16 = 150$ numbers in $[100, 999]$ having property A_6. Similarly, there are $\lfloor\frac{999}{7}\rfloor - \lfloor\frac{99}{7}\rfloor = 128$ numbers in $[100, 999]$ having property A_7. There are $\lfloor\frac{999}{42}\rfloor - \lfloor\frac{99}{42}\rfloor = 21$ numbers divisible by $\text{lcm}(6,7) = 42$, hence having both A_6 and A_7. Hence, the answer is $150 + 128 - 21 = 257$.

5. Suppose $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfies $\sum_{i=0}^n f(i) = n^2$, for all $n \in \mathbb{N}_0$. Find a closed form for $f(n)$.

   Consider the poset $\mathbb{N}_0$, with the usual order, and the usual $f(a,b) = f(b-a)$. We learned in class that $\mu(x,y) = \left\{\begin{matrix} 1 & y-x=0 \\ -1 & y-x=1 \\ 0 & y-x>1 \end{matrix}\right\}$. The problem specifies that $n^2 = (1 \star f)(0,n)$; hence $f = (n^2 \star \mu)(0,n) = \sum_{0 \le x \le n} x^2\mu(x,n) = \left\{\begin{matrix} -(n-1)^2 + n^2 & n>0 \\ 0 & n=0 \end{matrix}\right\} = \left\{\begin{matrix} 2n-1 & n>0 \\ 0 & n=0 \end{matrix}\right\}$.

Exam grades: High score=104, Median score=80, Low score=50